

REEXAMINATION OF $K\bar{K}$ THRESHOLD PHENOMENA WITH K^+K^- ATOM INCLUDED

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Abstract

We develop a general framework to study $K\bar{K}$ threshold phenomena with resonances f_0 , a_0 and K^+K^- atom included. Based on this formalism we predict that the production of the K^+K^- atom in $pd \rightarrow {}^3\text{He}X$ and similar reactions exhibits a drastic energy dependence due to the interplay with resonances f_0 (980) and a_0 (980). We point out that a set of few parameters describes $K\bar{K}$ threshold effects, f_0 and a_0 mesons, and K^+K^- atom. Our hope is that precision experimental study aimed at determining these parameters may shed more light on the nature of f_0 and a_0 resonances.

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1 Introduction

It is well known that there is an interesting sometimes controversial physics bearing upon the $K\bar{K}$ threshold phenomena and the nature of f_0 , and a_0 resonances. It is not the aim of our paper to give even a brief summary on this topic (see, e.g. Refs. [1-7]). Our goal is to argue that a valuable piece of new information on the subject may come from the study of an exotic K^+K^- atom (kaonium). The main point of this paper is that kaonium might be included into the effective Hamiltonian which describes $K\bar{K}$ threshold effects and resonances f_0, a_0 . Such a special approach to kaonium as compared to other exotic atoms is due to the fact that $K\bar{K}$ threshold region is completely overlapped by resonances f_0 and a_0 to both of which kaonium is coupled. Therefore examining kaonium we face the interference of several overlapping resonances. It is well known that such interference gives rise to various intriguing phenomena. In our case this will be the drastic energy behaviour of kaonium production amplitude. Our interest in describing kaonium furnished a major motivation of constructing the phenomenological effective Hamiltonian for the $K\bar{K}$ threshold region. Having constructed such a Hamiltonian we find ourselves in a position to study $K\bar{K}$ threshold anomalies and the mixing of f_0 and a_0 mesons due to the $K^0\bar{K}^0$ and K^+K^- thresholds splitting.

The organization of the paper is the following. The rest of this Section is devoted to introductory remarks on kaonium. Then in Section 2 we consider the specific reaction

$pd \rightarrow {}^3\text{He}X$ in which kaonium may be possibly observed. Our main formalism is presented in Section 3 where the effective Hamiltonian is constructed. In Section 4 we study the production of kaonium in its ground state and predict a drastic energy dependence of this process mentioned above. Section 5 is devoted to $K\bar{K}$ threshold anomalies. Section 6 contains the discussion about the mixing of f_0 and a_0 mesons. In Section 7 we present the set of parameters behind the effective Hamiltonian, and indicate those which have to be determined from future experiments. In Section 8 we estimate the mixing parameters of kaonium and resonances f_0 , a_0 making use of the alternative models for the f_0 and a_0 structure. Our main results are summarized in Conclusions.

Kaonium has not been observed yet but corresponding experiments are contemplated – see the paper [8] and references therein. To our knowledge the kaonium production was first discussed by Wycech and Green [8]. We have already mentioned that our study of kaonium is based on the idea of the interplay of kaonium and resonances f_0 and a_0 . This makes our approach (first outlined in [9]) different from that of [8]. We are not aware of any previous discussion of the mixing of an exotic atom and a hadron resonance.

Kaonium has the Bohr radius of $a_B = 2/\alpha m_{K^+} = 109.6 fm$, the ground state binding energy is $E_0 = \alpha^2 m_{K^+}/4 = 6.57 keV$. The expected decay width is $\Gamma = \alpha^3 m_{K^+}^2 ImA \simeq 640 eV$ [8], an alternative estimate for Γ will be given below. Kaonium is formed by electromagnetic interaction and its wave function has two isospin components with $I = 0$ and $I = 1$. Therefore it is mixed with both f_0 and a_0 ; different facets of this mixing will be discussed at length in what follows.

2 $pd \rightarrow {}^3\text{He}X$ reaction close to $K\bar{K}$ thresholds

To be specific we shall consider f_0, a_0 meson and kaonium production in the reaction

$$pd \rightarrow {}^3\text{He}X \rightarrow {}^3\text{He}\pi^+\pi^- \quad (1)$$

with the invariant mass of X being in the vicinity of K^+K^- and $K^0\bar{K}^0$ thresholds. The K^+K^- production threshold in reaction (1) corresponds to the kinetic energy of the incident proton beam equal to $T_p = 1.73 GeV$. A precision experimental study of the $K\bar{K}$ threshold region in the reaction (1) is a part of the program at SATURNE [10] and at COSY [11]. Although we shall discuss kaonium production only in the reaction (1), the formalism to be developed can be easily applied to kaonium production in other processes as well.

At this point a remark is in order. The term kaonium production used above is somewhat misleading. In what follows we shall explain in detail that the object X produced in reaction (1) close to K^+K^- threshold is a mixture of f_0 , a_0 mesons and kaonium atom. Therefore the question of what is the kaonium production probability in reaction (1) has to be replaced by the question of what is the admixture of kaonium in X and how this admixture manifests itself experimentally. The answer to the last question is one of the purposes of our paper.

We start with a standard equation for the cross section of reaction (1)

$$d\sigma = (2\pi)^4 \delta^{(4)}(P_p + P_d - P_{He} - P_{\pi^+} - P_{\pi^-}) \frac{|\mathcal{M}|^2}{4I} \prod_f \frac{d^3\mathbf{p}_f}{(2\pi)^3 2E_f}, \quad (2)$$

where the product runs over ${}^3\text{He}$, π^+ and π^- . In a typical experimental situation one is interested in the double differential cross section $d^2\sigma/dp_{He}d\Omega_{He}$ (here p_{He} stands for $|\mathbf{p}_{He}|$)

corresponding to the measurement of helium momentum and direction in the laboratory frame. Routine arguments allow to perform phase space integration with $|\mathcal{M}|^2$ factorized out. The result reads

$$\frac{d^2\sigma}{dp_{He}d\Omega_{He}} = |\mathcal{M}|^2 \frac{p_{He}^2}{2^8\pi^4 m_X^2 E_{He}} \frac{\lambda^{1/2}(m_X^2; m_\pi^2, m_\pi^2)}{\lambda^{1/2}(s; m_p^2, m_d^2)}, \quad (3)$$

where $s = (P_p + P_d)^2$, m_X is the invariant mass of X and $\lambda(x; y, z) = (x - y - z)^2 - 4yz$ is a kinematical function. In what follows use will be made of the T -matrix connected to the invariant amplitude via the relation

$$T = \mathcal{M} \prod_{i,f} (2E_i)^{-1/2} (2E_f)^{-1/2}, \quad (4)$$

where the initial state index i includes p and d , and the final state index f runs over 3He , π^+ , π^- . It is simply related to S -matrices as

$$S = 1 + i(2\pi)^4 \delta^{(4)}(P_f - P_i) T, \quad (5)$$

where P_f and P_i are the sums of the final and initial particles 4-momenta. In the dynamical model to follow the T -matrix will be considered as a function of the invariant mass m_X . Therefore Eqs.(3-4) should be supplemented by a kinematical relation between laboratory frame momentum of 3He and the invariant mass m_X . One defines θ as the angle between the direction of the incident proton and outgoing 3He . Then the following relation is obtained

$$m_X^2 = m_p^2 + m_d^2 + m_{He}^2 + 2E_p(m_d - E_{He}) - 2m_d E_{He} + 2P_p P_{He} \cos\theta. \quad (6)$$

According to (6) one can vary m_X either by changing the momentum of the incident proton with 3He momentum and angle kept fixed, or by measuring 3He spectrum with initial proton momentum kept fixed.

Our next goal is to get an expression for the T -matrix which corresponds to our physical picture in which f_0 and a_0 mesons dominate the $K\bar{K}$ threshold region and hence the kaonium production.

3 The T -matrix and the effective Hamiltonian

The literature on overlapping resonances, resonance "mixtures", and related topics is overwhelming. The formalism to be used here is based on two essentially equivalent approaches. The first one is the mass-matrix method which is very transparently presented in a paper by Kobzarev, Nikolaev and Okun [12]. An alternative way to get similar results was developed by Stodolsky [13] in what he called scattering theory of resonance "mixtures". We find it hardly necessary to retrieve here the basic equations. Rather we would like to concentrate on novel features which characterize the mixing of an atom and a resonance.

Consider reaction (1) with X being a mixture of f_0 , a_0 , and kaonium. The detailed composition of X will become clear below. Here we only mention that the mixing of f_0 and a_0 is possible via the coupling of both to kaonium. Then T -matrix, defined by Eqs.(4) and (5), satisfies the following equation

$$T = \langle \pi\pi | \hat{V} \hat{G} | A \rangle + T^0. \quad (7)$$

First we explain the notations in (7) and then consider each of its blocks in detail. The $|\pi\pi\rangle$ stands for the final ${}^3He\pi\pi$ system with definite momenta, \hat{V} is the transition operator corresponding to the $f_0 \rightarrow \pi\pi$ process (we remind that the decay $a_0 \rightarrow \pi\pi$ is forbidden). The Green's function \hat{G} describes f_0 - a_0 system plus 3He spectator and its coupling to resonance decay channels as well as to the kaonium. $|A\rangle$ is a (two component) creation amplitude of resonances f_0 , a_0 plus 3He spectator. Into the background term T^0 we have lumped direct ${}^3He\pi\pi$ production.

Now we consider the Green's function \hat{G} of mesons f_0 and a_0 (3He plays the role of spectator, i.e. it enters only through kinematics). We shall write \hat{G} as a 2×2 matrix in the basis corresponding to "bare" f_0 and a_0 , which we denote as $|f\rangle$ and $|a\rangle$. The term "bare" means that $|f\rangle$ and $|a\rangle$ are not mixed with decay channels and kaonium. These states are assumed to have definite isospin $I = 0$ for $|f\rangle$ and $I = 1$ for $|a\rangle$. Since we consider reaction (1) in the vicinity of $K\bar{K}$ thresholds the relative motion in the system $K\bar{K}$ can be treated nonrelativistically, which will be used in writing expressions for the corresponding propagators. All the above remarks on the operator \hat{G} allow to write it in the form

$$\hat{G} = \begin{pmatrix} G_{ff} & G_{fa} \\ G_{af} & G_{aa} \end{pmatrix} = (m_X - \hat{H} - \Delta\hat{H})^{-1}, \quad (8)$$

where m_X is the invariant mass of the system X in reaction (1), \hat{H} stands for the Hamiltonian of the bare $|f\rangle$ and $|a\rangle$

$$\hat{H} = \begin{pmatrix} E_f^{(0)} & 0 \\ 0 & E_a^{(0)} \end{pmatrix}, \quad (9)$$

while their coupling to decay channels $\pi\pi$, $\pi\eta$, K^+K^- , $K^0\bar{K}^0$ and kaonium is described by the matrix $\Delta\hat{H}$. It has the form

$$\Delta\hat{H} = \sum_{\alpha} \hat{V}^+ \frac{|\alpha\rangle\langle\alpha|}{m_X - m_{\alpha} + i0} \hat{V}, \quad (10)$$

where $|\alpha\rangle$ indicates any of the states ($\pi\pi$, $\pi\eta$, K^+K^- , $K^0\bar{K}^0$, kaonium) to which $|f\rangle$ and $|a\rangle$ are coupled, and m_{α} is its invariant mass. In the basis employed in Eq.(8) the transition operator \hat{V} is represented by a row (V_f, V_a) , while the f_0 and a_0 production amplitude A by a column $\begin{pmatrix} A_f \\ A_a \end{pmatrix}$. Now we must be reminded of the fact that due to the G -parity conservation only V_f element has nonvanishing projection on the final ${}^3He\pi\pi$ state. Therefore inserting Eq. (8) into (7), we get the following expression for the amplitude T :

$$T = V_{\pi\pi,f} G_{ff} A_f + V_{\pi\pi,f} G_{fa} A_a + T^0, \quad (11)$$

where

$$V_{\pi\pi,f} \equiv \langle \pi\pi | \hat{V} | f \rangle. \quad (12)$$

Equation (11) represents the gist of our approach. Now let us consider the contribution of the kaonium ground state into the T -matrix (11).

4 The amplitude behaviour near the kaonium ground state

In this section we shall discuss how the kaonium ground state manifests itself in the T -matrix (11) and in the cross section (3). To this end in Eq. (10) for $\Delta\hat{H}$ we single out from the sum the term corresponding to kaonium ground state, while the remainder add to \hat{H} given by Eq. (9). Such a regrouping yields

$$\hat{H} + \Delta\hat{H} = \frac{\hat{V}^+|at\rangle\langle at|\hat{V}}{m_X - m_{at}} + \begin{pmatrix} E_f - i\frac{\Gamma_f}{2} & 0 \\ 0 & E_a - i\frac{\Gamma_a}{2} \end{pmatrix}, \quad (13)$$

where the state $|at\rangle$ is the kaonium ground state with a pure Coulomb binding energy, i.e. $m_{at} = 2m_K - 6.57\text{keV}$. In the next Section we shall see that the second term of Eq. (13) displays only minor energy dependence within the m_X interval smaller than the distance between the ground and first excited states of kaonium. Therefore in this Section the quantities $E_f, E_a, \Gamma_f, \Gamma_a$ are considered as energy independent. The corrections to this approximation are essential for the cusp structure at K^+K^- threshold which will be considered in the next Section. We have also neglect the nondiagonal elements in the second term of (13). This amounts to neglecting the $f_0 - a_0$ mixing via kaonium excited states and, what is much more substantial, via the low energy parts of the K^+K^- and $K^0\bar{K}^0$ continuous spectra. The validity of such approximation will be discussed in the Section 6.

Now we substitute the effective Hamiltonian (13) into the Green's function (8) and T -matrix (11). After a few formal manipulations this yields

$$\begin{aligned} T = & \frac{V_{\pi\pi,f}A_f}{m_X - E_f + i\frac{\Gamma_f}{2}} + \\ & + \frac{1}{(m_X - E_{at} + i\frac{\Gamma_{at}}{2})} \frac{V_{\pi\pi,f}V_{f,at}^+}{(m_{at} - E_f + i\frac{\Gamma_f}{2})} \left[\frac{V_{at,f}A_f}{m_{at} - E_f + i\frac{\Gamma_f}{2}} + \right. \\ & \left. + \frac{V_{at,a}A_a}{m_{at} - E_a + i\frac{\Gamma_a}{2}} \right] + T^0, \end{aligned} \quad (14)$$

where $V_{at,r} = \langle at|\hat{V}|r\rangle$, $r = f, a$, and

$$E_{at} = m_{at} + \sum_{r=f,a} |V_{at,r}|^2 \frac{m_{at} - E_r}{(E_r - m_{at})^2 + \frac{\Gamma_r^2}{4}}, \quad (15)$$

$$\Gamma_{at} = \sum_{r=f,a} |V_{at,r}|^2 \frac{\Gamma_r}{(E_r - m_{at})^2 + \frac{\Gamma_r^2}{4}}. \quad (16)$$

Formulae (14-16) are the sought-for equations which display the effect of the kaonium ground state in reaction (1). They are reminiscent of equations describing giant nuclear resonances (see, e.g. [14-15]).

Equation (14) shows that as a function of the invariant mass m_X , the amplitude T in the reaction (1) has the pole at $m_X = E_{at} - i\Gamma_{at}/2$. It corresponds to the atomic level which has acquired the shift (15) and the width (16) due to the mixing of f_0 and a_0 mesons.

The first term in Eq.(14) looks as if $T(m_X)$ had another pole at $m_X = E_f - i\Gamma_f/2$ corresponding to the f_0 resonance. The issue, however, is more subtle. The simple pole structure for f_0 is invalidated by the energy dependence of its width Γ_f and the existence of the so-called "shadow" poles on different sheets of the energy Riemann surface [3,4,16]. We shall return to this point in Section 6, for more discussion see the cited references.

Equation (14) may be presented in a more concise and transparent form by introducing the following "propagators"

$$g_f(m) = \frac{1}{m - E_f + i\frac{\Gamma_f}{2}}, \quad g_a(m) = \frac{1}{m - E_a + i\frac{\Gamma_a}{2}}, \quad g_{at}(m) = \frac{1}{m - E_{at} + i\frac{\Gamma_{at}}{2}} \quad (17)$$

Then (14) takes the form

$$T = V_{\pi\pi,f}g_f(m_X)A_f + V_{\pi\pi,f}g_f(m_{at})V_{f,at}^+g_{at}(m_X)[V_{at,f}g_f(m_{at})A_f + V_{at,a}g_a(m_{at})A_a] + T^0. \quad (18)$$

The first and the second terms of Eq.(18) may be represented by diagrams in Figs. 1(a) and 1(b) correspondingly.

In order to display the behaviour of the cross section (3) in the vicinity of the kaonium ground state we introduce the quantities

$$\Delta E_{at}^{(r)} = V_{at,r}g_r(m_{at})V_{r,at}^+ = \frac{|V_{at,r}|^2}{m_{at} - E_r + i\frac{\Gamma_r}{2}}, \quad r = f, a. \quad (19)$$

which have a transparent interpretation of the complex energy shift of the atomic level due to its mixing with a resonance r . In fact, Eqs. (15-16) yield

$$E_{at} - i\frac{\Gamma_{at}}{2} = m_{at} + \Delta E_{at}^{(f)} + \Delta E_{at}^{(a)}. \quad (20)$$

In these notations the amplitude T (18) reads

$$T = V_{\pi\pi,f}g_f(m_{at})\left\{1 + \frac{\Delta E_{at}^{(f)} + \eta\Delta E_{at}^{(a)}}{m_X - m_{at} - \Delta E_{at}^{(f)} - \Delta E_{at}^{(a)}}\right\}A_f + T^0, \quad (21)$$

where

$$\eta = \frac{V_{f,at}^+A_a}{V_{a,at}^+A_f}. \quad (22)$$

A careful look at Eqs. (21-22) leads to striking observation. Suppose for a moment that the background term T^0 as well as the coupling to a_0 meson are absent ($T^0 = 0$, $\Delta E_{at}^{(a)} = 0$). Then the amplitude T has a zero at $m_X = m_{at}$ (compare to the well-known "dipole" phenomena considered in chapter 8 of [17]). The structure above is similar to what is called "Fano zero" [18] of the amplitude.

The general case corresponds to a finite value of the background term T^0 and a certain (complex) value of the parameter η . Unless the background becomes dominant, the cross section still has a dip – bump structure at the vicinity of the atomic level. The exact position of the minimum as well as the height of the adjacent peak depends upon the parameter η . In particular from Eq.(21) one notices that for $\eta = 1$ the first term of (21) again has a zero at the same point $m_X = m_{at}$.

From Eqs. (19), (21) it follows that the energy scale of the zero-peak (or dip-bump) structure is governed by the mixing parameter $|V_{r,at}|^2$, $r = f, a$. This quantity, which is sensitive to the nature of f_0, a_0 mesons, is discussed in Sections 7, 8.

In Fig. 2 we plot the typical cross section $d^2\sigma/dp_{He}d\Omega_{He}$ (see Eq. (3)) as a function of m_X . We remind that m_X is connected to 3He momentum and angle via relation (6). The set of parameters behind Fig. 2 is discussed in Section 7.

5 $K\bar{K}$ threshold effects

In this section we give an account of anomalies which show up at $K\bar{K}$ thresholds. The K^+K^- and $K^0\bar{K}^0$ thresholds are splitted by 8 MeV, and due to the Coulomb attraction in the K^+K^- channel the cusp phenomena are quite different. First we consider the case of charged kaons.

The general character of threshold anomaly in the case of Coulomb attraction is well known. As the energy of the $\pi^+\pi^-$ system in reaction (1) approaches the K^+K^- threshold from below, we will observe a series of ever more rapid oscillations due to the excitation of the infinitely numerous Coulomb levels of kaonium. The limit of the cross section (3) below threshold does not exist since the threshold energy is the accumulation point for these resonances. The purpose of this section is to reproduce these general results from our equations and to investigate which of the parameters of our model may be determined from precision measurements of the K^+K^- threshold effects.

Again we start with regrouping different terms of the effective Hamiltonian $\hat{H} + \Delta\hat{H}$ given by Eqs. (9-10). This time we single out from $\Delta\hat{H}$ (10) all contributions stemming from S -wave states of the K^+K^- system (atomic levels plus continuous spectrum). The transition operator \hat{V} (see Eq. (10) and the text afterwards) is of short-range character and hence dominated by high momentum components. The K^+K^- system near threshold is on the contrary characterized by low momenta. Therefore one has

$$\hat{V}^+|\alpha\rangle = \int \frac{d^3p}{(2\pi)^3} \hat{V}^+|\vec{p}\rangle \langle \vec{p}|\alpha\rangle \simeq \hat{V}^+|K^+K^-\rangle \psi_\alpha(0), \quad (23)$$

where use was made of the fact that the main contribution to the integral comes from small momenta typical for $\langle \vec{p}|\alpha\rangle$, at these momenta $\hat{V}^+|\vec{p}\rangle$ can be taken out of the integral as a momentum independent vector $\hat{V}^+|K^+K^-\rangle$, and the remaining integral yields the K^+K^- system wave function at the origin $\psi_\alpha(0)$.

In particular, the parameter $V_{at,r} = \langle at|\hat{V}|r\rangle$ ($r = f, a$), which characterizes the mixing of kaonium ground state with the corresponding mesons, takes the form

$$V_{at,r} = \psi_{at}(0) \langle K^+K^-|\hat{V}|r\rangle = \sqrt{\frac{\alpha^3 m_K^3}{8\pi}} \langle K^+K^-|\hat{V}|r\rangle \quad (24)$$

The geometrical factor has been singled out, and the remaining "reduced" matrix element $\langle K^+K^-|V|r\rangle$ depends on the nature of f_0, a_0 mesons. It will be discussed in Section 8.

Summation in Eq.(10) over K^+K^- states implies both integration over continuous spec-

trum and summation over atomic levels. Therefore we have

$$\Delta\hat{H} = \Delta\hat{H}' + \hat{V}^+ |K^+ K^- \rangle \langle K^+ K^-| \hat{V} \left[\int \frac{d^3p}{(2\pi)^3} \frac{|\psi_p(0)|^2}{\Delta m - \frac{p^2}{m_{K^+}} + i0} + \sum_{n=1}^{\infty} \frac{|\psi_n(0)|^2}{\Delta m + \frac{m_K \alpha^2}{4n^2}} \right], \quad (25)$$

where $\Delta m = m_X - 2m_{K^+}$, and

$$|\psi_n(0)|^2 = \frac{\alpha^3 m_K^3}{8\pi n^3}, \quad |\psi_p(0)|^2 = \frac{\pi \alpha m_K}{p \{1 - \exp(-\frac{\pi \alpha m_K}{p})\}}. \quad (26)$$

The remaining part $\Delta\hat{H}'$ in Eq.(25) arises from the f_0 and a_0 decay channels other than $K^+ K^-$ (i.e. $\pi\pi, \pi\eta, K^0 \bar{K}^0$).

The integral in (25) diverges at large values of p . This is a formal difficulty since at large momenta the operator \hat{V} induces cutoff (also at large values of p one has to use relativistic kinematics). Within the energy interval $|\Delta m| = |m_X - 2m_{K^+}| \leq E_0 = \frac{\alpha^2 m_{K^+}}{4}$, where Coulomb effects are important, Eq.(25) yields

$$\Delta\hat{H} = \Delta\hat{H}' + \hat{V}^+ |K^+ K^- \rangle \langle K^+ K^-| \hat{V} \left[\Phi + \frac{\alpha m_{K^+}^2}{4} \begin{cases} \cot(\pi \sqrt{\frac{m_{K^+} \alpha^2}{4|\Delta m|}}), & \Delta m < 0 \\ -i, & \Delta m > 0 \end{cases} \right], \quad (27)$$

where Φ is a real function which experiences smooth variations not exceeding αm_K^2 over the interval $|\Delta m| \leq E_0$.

The Equations (8),(11),(27) retrieve the Coulomb threshold anomaly as it was outlined at the beginning of this Section. Eq.(27) also proves the validity of the approximation (13) with constant E_r and $\Gamma_r(r = f, a)$ at the immediate vicinity of the kaonium ground state.

Now we briefly discuss what happens at the $K^0 \bar{K}^0$ threshold. Here the discrete spectrum is absent, and instead of Eq. (26) we have $|\psi_p(0)|^2 = 1$. Equation (25) takes the form

$$\Delta\hat{H} = \Delta\hat{H}'' + \hat{V}^+ |K^0 \bar{K}^0 \rangle \langle K^0 \bar{K}^0| \hat{V} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\Delta m - \frac{p^2}{m_{K^0}} + i0}, \quad (28)$$

with $\Delta m = m_X - 2m_{K^0}$ and the same remarks applied concerning the convergency of the integral. The energy dependent part of (28) reads

$$\Delta\hat{H}^{(n)} = \hat{V}^+ |K^0 \bar{K}^0 \rangle \langle K^0 \bar{K}^0| \hat{V} \begin{cases} \frac{m_{K^0}^2}{4\pi} \sqrt{\frac{|\Delta m|}{m_{K^0}}}, & \Delta m < 0 \\ -i \frac{m_{K^0}^2}{4\pi} \sqrt{\frac{|\Delta m|}{m_{K^0}}}, & \Delta m > 0. \end{cases} \quad (29)$$

Thus at $K^0 \bar{K}^0$ threshold the amplitude displays the standard behaviour of the $\sqrt{2m_{K^0} - m_X}$ type. The same pattern also restores in the $K^+ K^-$ channel when $|m_X - 2m_{K^+}| \gg E_0$, i.e. away of the narrow Coulomb region. Eq. (29) and its analogue for charged kaons will be the starting points of the next Section.

6 Mixing of the f_0 and a_0 mesons

Now we proceed to the energy scale of the order of the interval between $K^+ K^-$ and $K^0 \bar{K}^0$ thresholds: $2m_{K^0} - 2m_{K^+} = 8MeV$. It by far exceeds a characteristic Coulomb energy

$E_0 = 6.6 \text{ keV}$ therefore we disregard Coulomb effects in the present section. Then the energy dependence of the effective Hamiltonian $\hat{H} + \Delta\hat{H}$ (9-10) is given by the two terms of the form (29) corresponding to the charged and neutral kaon channels

$$\begin{aligned} \hat{H} + \Delta\hat{H} &= \hat{H}^{(0)} + \Delta\hat{H}^{(ch)}(m_X) + \Delta\hat{H}^{(n)}(m_X) = \\ &= \hat{H}^{(0)} + \hat{V}^+ |K^+ K^- \rangle \langle K^+ K^-| \hat{V} (-i \frac{m_K^2}{4\pi}) \sqrt{\frac{m_X - 2m_{K^+}}{m_K}} + i0 + \\ &\quad + \hat{V}^+ |K^0 \bar{K}^0 \rangle \langle K^0 \bar{K}^0| \hat{V} (-i \frac{m_K^2}{4\pi}) \sqrt{\frac{m_X - 2m_{K^0}}{m_K}} + i0, \end{aligned} \quad (30)$$

where adding $(+i0)$ means that for $m_X < 2m_K$ the square roots acquire positive imaginary parts, and where the $K^+ - K^0$ mass difference is ignored except for the under square root expressions. The first term $\hat{H}^{(0)}$ of Eq.(30) includes the Hamiltonian \hat{H} (7) of the bare $|f \rangle$ and $|a \rangle$ states (see Eq.(9)) plus contributions from f_0 , a_0 decay channels other than $K\bar{K}$ (such as $\pi\pi$ and $\pi\eta$).

We now remind that the states $|f \rangle$ and $|a \rangle$, which were introduced in the Sec. 3, have definite isospins $I_f = 0$, $I_a = 1$. Then isospin invariance implies

$$\langle K^+ K^- | \hat{V} | f \rangle = \langle K^0 \bar{K}^0 | \hat{V} | f \rangle, \quad \langle K^+ K^- | \hat{V} | a \rangle = - \langle K^0 \bar{K}^0 | \hat{V} | a \rangle. \quad (31)$$

Having got these relationships one can put the Eq. (30) into a more instructive form. For this purpose we introduce the notations

$$D = | \langle K^+ K^- | \hat{V} | f \rangle |^2 \frac{m_K^2}{4\pi}, \quad (32)$$

$$\zeta = \frac{\langle K^+ K^- | \hat{V} | a \rangle}{\langle K^+ K^- | \hat{V} | f \rangle}. \quad (33)$$

Then

$$\begin{aligned} \hat{H} + \Delta\hat{H} &= \begin{pmatrix} E'_f - i\Gamma_f/2 & 0 \\ 0 & E'_a - i\Gamma_a/2 \end{pmatrix} + \\ &+ D \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & |\zeta|^2 \end{pmatrix} (-i) \sqrt{\frac{m_X - 2m_{K^+}}{m_K}} + i0 + D \begin{pmatrix} 1 & -\zeta \\ -\bar{\zeta} & |\zeta|^2 \end{pmatrix} (-i) \sqrt{\frac{m_X - 2m_{K^0}}{m_K}} + i0. \end{aligned} \quad (34)$$

In this equation the first term is just a parameterization for $\hat{H}^{(0)}$ in Eq.(30). It must be diagonal for all its contributions, discussed after Eq.(30), have only diagonal elements.

From Eqs. (34),(8) and (11) it is clear that in the $K\bar{K}$ threshold region the cross section (3) (again considered as a function of m_X) is influenced by both mesons f_0 and a_0 and also depends on the parameter ζ . Some typical plots are presented in Fig. 3.

Far away of the $K\bar{K}$ thresholds the nondiagonal elements in the second and third terms on the right hand side of Eq.(34) cancel each other. However in the threshold region they are quite substantial. In fact, if m_X is fixed exactly at the $K^+ K^-$ threshold so that the second term in (34) vanishes, the nondiagonal contribution due to the third term amounts to $-\zeta D \sqrt{2(m_{K^0} - m_{K^+})/m_K} = -27\zeta \text{ MeV}$ which might be compared to $\Gamma_f/2 = 108 \text{ MeV}$ (numerical estimates for D and Γ_f follow Ref. [4], see Section 7 for more details).

At this point we return to Eq.(13) describing the effective Hamiltonian in the vicinity of kaonium ground state and address the question of its validity. To this end we make use of (34) and introduce into the second term of (13) the contributions hitherto omitted. To be more explicit we put all quantities in MeV units. Then the second term of (13) takes the form

$$\begin{pmatrix} E'_f - i\Gamma_f/2 + 27 & -27\zeta \\ -27\bar{\zeta} & E'_a - i\Gamma_a/2 + 27|\zeta|^2 \end{pmatrix}. \quad (35)$$

Therefore our approximation (13) with vanishing nondiagonal elements in the second term is appropriate provided $f_0 K\bar{K}$ coupling constant is not smaller than the coupling constant $a_0 K\bar{K}$. To our knowledge this question is still ambiguous and controversial [16].

Introduction of nondiagonal elements in the last term of Eq.(13) does not change the general conclusions of Sec.4 concerning a narrow resonance at the kaonium level but corresponding formulas will be well complicated. For the complex energy shift of the kaonium ground level (20), the expansion in a power series of the parameter ζ gives the following linear term

$$\begin{aligned} & -2 \frac{\text{Re}(\zeta V_{at,a}^* V_{at,f})}{(m_{at} - E_f + i\Gamma_f/2)(m_{at} - E_a + i\Gamma_a/2)} \cdot 27 \text{MeV} = \\ & - \frac{|\psi_{at}(0)|^2 |< K^+ K^- | \hat{V} | a >|^2 |< K^+ K^- | \hat{V} | f >|^2}{2\pi(m_{at} - E_f + i\Gamma_f/2)(m_{at} - E_a + i\Gamma_a/2)} \sqrt{2m_K^3(m_{K^0} - m_{K^+})} \end{aligned} \quad (36)$$

7 Parameters of the model

In our description of f_0 and a_0 mesons we have the following set of parameters to be determined: five real quantities $E'_f, E'_a, \Gamma_f, \Gamma_a, D$ and a complex quantity ζ ; they are presented by Eqs. (32-34). In addition we have two production amplitudes A_f, A_a (see Eq.(11)) and the transition matrix element $V_{\pi\pi,f}$ (see (11-12)).

The other quantities in our formalism are related to those nine. Namely, $< K\bar{K} | \hat{V} | r >$ ($K = K^+, K^0$; $r = f, a$) can be determined from Eqs. (32-33) while its bearing on the nature of f_0, a_0 mesons will be the subject of the next Section. Parameters E_f and E_a in Eq. (13) are connected to E'_f and E'_a via Eqs. (13),(34) as

$$E_f = E'_f + D \sqrt{\frac{2m_{K^0} - 2m_{K^+}}{m_K}} \quad (37)$$

$$E_a = E'_a + |\zeta|^2 D \sqrt{\frac{2m_{K^0} - 2m_{K^+}}{m_K}}$$

To obtain the values of E'_f, Γ_f and D we remind that for m_X well outside the 8 MeV $K^+ K^-, K^0 \bar{K}^0$ interthreshold region the off-diagonal elements of the effective Hamiltonian $\hat{H} + \Delta\hat{H}$ (34) are cancelled, so that f_0 and a_0 are not mixed. Then $\pi\pi$ production in the reaction (1) goes only through f_0 meson ($a_0 \rightarrow \pi\pi$ is G- parity forbidden). Thus the Green's function \hat{G} (8) becomes diagonal and the amplitude (11) takes the simple form

$$T = \frac{V_{\pi\pi,f} A_f}{m_X - E'_f + i\Gamma_f/2 + 2iD \sqrt{\frac{m_X - 2m_K}{m_K}} + i0} + T^0, \quad (38)$$

which is just a Breit–Wigner amplitude with mass–dependent width. Recent analysis of Breit–Wigner contribution to the f_0 structure is presented in Refs. [3,4]. In our present paper we stick to the set of parameters proposed by Zou and Bugg [4]; our set of parameters is related to their via

$$\begin{aligned} E'_f &= \frac{M_R^2}{2m_K} = 916 MeV \\ \Gamma_f &= \frac{g_1}{m_K} \sqrt{1 - \frac{m_\pi^2}{m_K^2}} = 214 MeV \\ D &= \frac{g_2}{4m_K} = 213 MeV \end{aligned} \tag{39}$$

where M_R, g_1, g_2 are quantities used in [4]. This yields $E_f = 943$ MeV in Eq. (A1) and $\Delta E_{at}^{(f)} = 66 - 162i$ eV in Eq. (17).

As far as the parameters E'_a , Γ_a and ζ are concerned, we are not aware of the $a_0(980)$ pole structure analysis which could provide their values.

8 Mixing parameter as a probe to the nature of the f_0, a_0 mesons

In Section 4 we have shown that the energy dependence of the zero-peak (dip–bump) structure due to kaonium is governed by the quantities $|V_{at,r}|^2$, $r = f, a$. This is where the nature of the mesons comes into play. The factorization (24)

$$|V_{at,r}|^2 = |\psi_{at}(0)|^2 < K^+ K^- | V | r >^2 \tag{40}$$

shows that the quantity $|V_{at,r}|^2$ is proportional to the absolute value square of the ”reduced” mixing parameter $< K^+ K^- | V | r >$. The geometrical factor $|\psi_{at}(0)|^2$ characterizes an overlap of the wave functions.

In order to be transparent and avoid cumbersome equations, we consider a toy model with only one meson, say f_0 , and try to estimate the quantity $|V_{at,f}|^2$ for different hypotheses on the nature of this meson [1–5].

Consider first an interpretation of the f_0 as a $K\bar{K}$ molecule, i.e. a deuteron–like state. A simple estimate yields

$$| < K^+ K^- | V | f_0 > |^2 \simeq \frac{8\pi}{m_K} \left(\frac{|\varepsilon - 2m_K|}{m_K} \right)^{1/2} = 1.1 \cdot 10^{-2} MeV^{-1} \tag{41}$$

where $\varepsilon = 988 - 23i$ MeV is the second–sheet pole for $f_0(980)$ [4]. This estimate corresponds to numerical value $\Gamma_{at} \simeq 330$ eV in Eq.(16).

For the alternative compositions of f_0 (e.g. $q\bar{q}$, multiquark, glueball, etc.), the natural way to parameterize $< K^+ K^- | V | f_0 >$ is through the Jaffe–Low P –matrix [19].

In P –matrix terminology f_0 is the P –matrix ”primitive” with eigenvalue E_n , radius b and the coupling to hadronic channels given by a residue λ_n . In terms of these quantities the P –matrix reads:

$$P = k \cot(kb + \delta) = P_0 + \frac{\lambda_n}{E - E_n} \tag{42}$$

Straightforward but somewhat lengthy calculations [20,21] lead to the estimate

$$| \langle K^+ K^- | V | f_0 \rangle |^2 = \frac{8\pi}{m_k} \lambda_n b^2 \quad (43)$$

The value of the residue λ_n of f_0 into $K^+ K^-$ channel is subject to large uncertainties. As an educated guess we can take the value of λ_n for $q^2 \bar{q}^2$ states in 1 GeV mass region from [22]. This yields $\Gamma_{at} \sim 160 \text{ eV}$ for $b = 0.8 fm$ and $\Gamma_{at} \sim 8 \text{ eV}$ for $b = 0.2 fm$ as suggested in [5].

9 Conclusions

We have seen that our model allows on the one hand to include kaonium into the realm of the $K\bar{K}$ threshold phenomena, and on the other permits to reexamine the whole problem from somewhat new point of view. We have predicted that the interplay of kaonium and f_0, a_0 mesons results in drastic behaviour of the amplitude in the vicinity of kaonium ground state. The observation of this fascinating structure calls for high precision experiments and we are not in a position to comment on their feasibility in the foreseeable future.

The effective Hamiltonian we have constructed has provided us with interesting insights on the connections between various physical quantities: meson positions and widths, threshold cusps, isospin breaking effects, kaonium manifestation. We have argued that the $f_0 - a_0$ mixing definitely influences the cross section of reaction (1) in the threshold region.

In future work it is possible to introduce more sophisticated topology of the poles ("shadow poles"), to discuss in more detail the $f_0 - a_0$ mixing in view of various predictions on the corresponding coupling constants. Another point to improve is to be more specific concerning the background amplitude T_0 ; to this end experimental information on the excitation curve around the $K\bar{K}$ threshold is needed. With T_0 really taken into consideration one can use the standard machinery [23] to unitarize the complete amplitude.

Our hope is that we have presented additional arguments in favour of precision studies of the $K\bar{K}$ threshold region.

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Figure Captions

FIG. 1. The diagrams representing the resonance terms in Eq.(18) for the $\pi\pi$ production amplitude when the invariant mass of the $\pi\pi$ system is close to the energy of the K^+K^- bound state.

FIG. 2. The dip-bump structure of the cross section at the vicinity of the kaonium ground level. This illustration corresponds to the amplitude given by Eq.(21) with the parameters $V_{at,r} = 0.142 \text{ MeV}$ ($r = f, a$), $E_r = 940 \text{ MeV}$, $\Gamma_r = 214 \text{ MeV}$, $\eta = 0.5i$, $T^0 = 0$.

FIG. 3. Shape of the resonance contribution to the amplitude of the reaction $pd \rightarrow {}^3\text{He}\pi\pi$. The background amplitude T^0 is taken to be zero. We use the effective Hamiltonian (34) with the following set of parameters: $E'_f = E'_a = 916 \text{ MeV}$, $\Gamma_f = \Gamma_a = 214 \text{ MeV}$, $D = 213 \text{ MeV}$ and equal f_0 and a_0 production amplitudes $A_f = A_a$. Solid line corresponds to the similar contributions from f_0 and a_0 mesons ($\zeta = 1$) and the dashed line to the f_0 dominance ($\zeta = 0$).